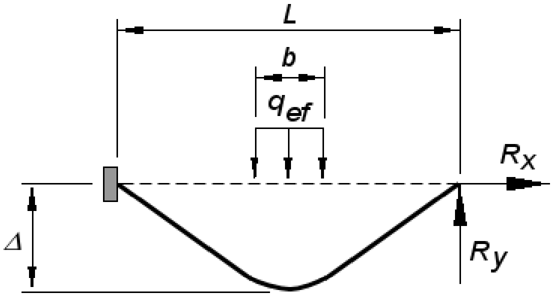
SECTION C

DERIVATION OF EQUATIONS FOR CODE REQUIRED LOADING ON INTERMEDIATE RAILING COMPONENTS

C.1—LOAD-DEFLECTION RELATIONSHIP FOR AN EXTENSIBLE, FLEXIBLE CABLE UNDER A PARTIAL UNIFORM LOAD



Symbols and Notations

Δ	Deflection of cable under uniform load, q_{ef}
b	Length of partial uniform load, in.
L	Spacing between intermediate supports, in.
q_{ef}	Partial uniform load required to produce deflection Δ, plf.
R_x	In-plane end reaction, due to deflection of cable, lbs.
R_y	Out-of-plane end reaction, due to deflection of cable, lbs.
s	Length of the curved segment of cable under the partial uniform load, in.
s_1	Length of the straight segment of cable between the partial uniform load and the support, in.
T_o	Tension load in cable at point of maximum deflection, lbs.
T_1	Tension load in straight segment of cable, lbs.
T_{avg}	Average tension load in curved segment of cable, lbs.

Objective

Given a mid-span deflection, Δ, determine the partial uniform load required to produce that deflection.

Determination of Reactions

Out-of-plane end reactions, $R_{y,}$ can be calculated by taking the sum of moments about one of the support points:

$$\Sigma M = 0: \qquad q_{ef} \cdot b \cdot \frac{L}{2} - R_y \cdot L = 0$$

$$R_y \cdot L = \frac{q_{ef} \cdot b \cdot L}{2}$$

$$R_y = \frac{q_{ef}\cdot b}{2} \tag{1}$$

The in-plane end reaction, *R*$_{x,}$ can be calculated by taking the sum of the moments about the mid-point of the cable, using forces to the right of the applied load:

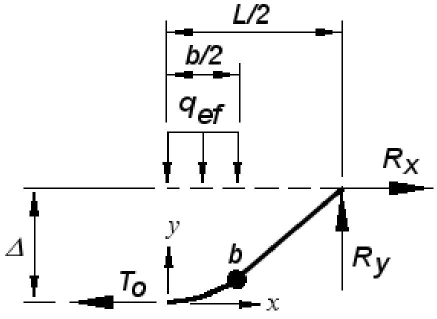
$$\Sigma\, M = 0: \qquad q_{ef}\cdot\frac{b}{2}\cdot\frac{L}{4} + R_x\cdot\Delta - R_y\cdot\frac{L}{2} = 0$$

$$R_x\cdot\Delta = \frac{R_y\cdot L}{2} - \frac{q_{ef}\cdot b\cdot L}{8}$$

$$R_x = \frac{\frac{q_{ef}\cdot b}{2}\cdot\frac{L}{2} - \frac{q_{ef}\cdot b\cdot L}{8}}{\Delta} = \frac{2\cdot q_{ef}\cdot b\cdot L}{8\cdot\Delta} - \frac{q_{ef}\cdot b\cdot L}{8\cdot\Delta}$$

$$R_x = \frac{q_{ef}\cdot b\cdot L}{8\cdot\Delta} \tag{2}$$

Strain Compatibility



The deflected shape of the cable under the uniform load is a parabola of the form:

$$y = \frac{q\cdot x^2}{2T_o}$$

where the origin (*x*=0 and *y*=0) is the point of maximum deflection.

The mid-span tension in the cable, *T*$_{o}$, can be calculated by taking the sum of the moments about the right-hand end point of the cable, using forces to the right of the applied load:

$$\Sigma\, M = 0: \qquad q_{ef}\cdot\frac{b}{2}\cdot\frac{L}{4} - T_o\cdot\Delta = 0$$

$$T_o = \frac{q_{ef}\cdot b\cdot L}{8\cdot\Delta} \tag{3}$$

which is equal to *R*$_{x}$, obtained in Eqn. (2).

Substituting Eqn. (3) into the parabola equation gives us:

$$y = \frac{q_{ef}\cdot x^2}{2\cdot\left(\frac{q_{ef}\cdot b\cdot L}{8\cdot\Delta}\right)}$$

$$y = \frac{4\cdot\Delta\cdot x^2}{b\cdot L} \qquad \textbf{(4)}$$

The length of the parabola from midpoint to the end of the partial uniform load is given by:

$$s = \int_0^{\frac{b}{2}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \textbf{(5)}$$

where *dy/dx* represents the incremental change in *y*, given an incremental change in *x*, which is the derivative of Eqn (4):

$$\frac{dy}{dx} = \frac{d}{dx}\cdot\left(\frac{4\cdot\Delta\cdot x^2}{b\cdot L}\right)$$

$$\frac{dy}{dx} = \frac{8\cdot\Delta\cdot x}{b\cdot L} \qquad \textbf{(6)}$$

Substituting Eqn. (6) back into Eqn. (5) yields:

$$s = \int_0^{\frac{b}{2}} \sqrt{1 + \left(\frac{8\cdot\Delta\cdot x}{b\cdot L}\right)^2}\, dx$$

The cable deflection at the point where the partial uniform load ends can be calculated using Eqn. (4):

$$\Delta_b = \Delta - y_b$$

$$\Delta_b = \Delta - \frac{4\cdot\Delta\cdot b^2}{b\cdot L}$$

$$\Delta_b = \Delta - \frac{4\cdot\Delta\cdot b}{L}$$

The length of the straight segment of cable between the uniform load and the support can be calculated using the Pythagorean theorum:

$$s_1 = \sqrt{{\Delta_b}^2 + \left(\frac{L-b}{2}\right)^2}$$

$$s_1 = \sqrt{\left(\Delta - \frac{4\cdot\Delta\cdot b}{L}\right)^2 + \left(\frac{L-b}{2}\right)^2}$$

The cable extension, δ, over the original length, *L*, is:

$$\delta = 2\cdot(s + s_1) - L$$

$$\delta = 2\cdot\left[\int_0^{\frac{b}{2}}\sqrt{1+\left(\frac{8\cdot\Delta\cdot x}{b\cdot L}\right)^2}\,dx + \sqrt{\left(\Delta-\frac{4\cdot\Delta\cdot b}{L}\right)^2+\left(\frac{L-b}{2}\right)^2}\right]-L \qquad \textbf{(7)}$$

The tension in the straight segment of cable, between the end of the uniform load and the support, T_1, can also be computed using the Pythagorean theorum:

$$T_1 = \sqrt{R_x^{\,2}+R_y^{\,2}}$$

Substituting R_y and R_x from Eqns (1) and (2) yields:

$$T_1 = \sqrt{\left(\frac{q_{ef}\cdot b\cdot L}{8\cdot\Delta}\right)^2+\left(\frac{q_{ef}\cdot b}{2}\right)^2}$$

$$T_1 = \frac{q_{ef}\cdot b}{2}\cdot\sqrt{\left(\frac{L}{4\cdot\Delta}\right)^2+1} \qquad \textbf{(8)}$$

To determine an equation for tension in the cable at any point along the loaded parabola, we turn once again to the Pythagorean theorum:

$$T = \sqrt{R_x^{\,2}+R_y^{\,2}}$$

Recall that T_o is equal to R_x, therefore this equation becomes:

$$T = \sqrt{T_o^{\,2}+\left(\frac{q_{ef}\cdot b}{2}\right)^2}$$

Substituting x for $b/2$, gives us an equation for tension in the cable at any point along the loaded parabola:

$$T(x) = \sqrt{T_o^{\,2}+\left(q_{ef}\cdot x\right)^2} \qquad \textbf{(9)}$$

The average tension in the cable along the parabola can be obtained by integrating Eqn. (9) and then dividing by the original length of the segment, $b/2$:

$$T_{avg} = \frac{\displaystyle\int_0^{\frac{b}{2}}\sqrt{T_o^{\,2}+\left(q_{ef}\cdot x\right)^2}\,dx}{\frac{b}{2}}$$

Substituting Eqn. (3) into the above equation and rearranging yields:

$$T_{avg} = \frac{2}{b}\cdot\int_0^{\frac{b}{2}}\sqrt{\left(\frac{q_{ef}\cdot b\cdot L}{8\cdot\Delta}\right)^2+\left(q_{ef}\cdot x\right)^2}\,dx$$

$$T_{avg} = \frac{2 \cdot q_{ef}}{b} \cdot \int_0^{\frac{b}{2}} \sqrt{\left(\frac{b \cdot L}{8 \cdot \Delta}\right)^2 + x^2}\, dx$$

The total elongation is a result of the cable stretch due to the tension in the cable over two distinct regions, the parabolic segment under the load and the straight segments between the load and the supports. The total elongation is:

$$\delta = \frac{T_{avg} \cdot b}{E \cdot A} + \frac{T_1 \cdot (L - b)}{E \cdot A}$$

$$\delta = \frac{b}{E \cdot A} \cdot \frac{2 \cdot q_{ef}}{b} \cdot \int_0^{\frac{b}{2}} \sqrt{\left(\frac{b \cdot L}{8 \cdot \Delta}\right)^2 + x^2}\, dx + \frac{L - b}{E \cdot A} \cdot \left[\frac{q_{ef} \cdot b}{2} \cdot \sqrt{\left(\frac{L}{4 \cdot \Delta}\right)^2 + 1}\right]$$

$$\delta = \frac{q_{ef}}{E \cdot A} \cdot \left[2 \cdot \int_0^{\frac{b}{2}} \sqrt{\left(\frac{b \cdot L}{8 \cdot \Delta}\right)^2 + x^2}\, dx + \frac{(L - b) \cdot b}{2} \cdot \sqrt{\left(\frac{L}{4 \cdot \Delta}\right)^2 + 1}\right] \qquad \textbf{(10)}$$

Simultaneous Equations

Since the total elongation given by Eqns. (7) and (10) must be the same, we now have two equations which can be used to solve for one variable in terms of the other.

Substituting Eqn. (7) for δ in Eqn. (10) and rearranging to solve for q_{ef} gives us:

$$2 \cdot \left[\int_0^{\frac{b}{2}} \sqrt{1 + \left(\frac{8 \cdot \Delta \cdot x}{b \cdot L}\right)^2}\, dx \ldots + \sqrt{\left(\Delta - \frac{4 \cdot \Delta \cdot b}{L}\right)^2 + \left(\frac{L - b}{2}\right)^2}\right] - L = \frac{q_{ef}}{E \cdot A} \cdot \left[2 \cdot \int_0^{\frac{b}{2}} \sqrt{\left(\frac{b \cdot L}{8 \cdot \Delta}\right)^2 + x^2}\, dx \ldots + \frac{(L - b) \cdot b}{2} \cdot \sqrt{\left(\frac{L}{4 \cdot \Delta}\right)^2 + 1}\right]$$

$$q_{ef} = E \cdot A \cdot \frac{2 \cdot \left[\int_0^{\frac{b}{2}} \sqrt{1 + \left(\frac{8 \cdot \Delta \cdot x}{b \cdot L}\right)^2}\, dx + \sqrt{\left(\Delta - \frac{4 \cdot \Delta \cdot b}{L}\right)^2 + \left(\frac{L - b}{2}\right)^2}\right] - L}{2 \cdot \int_0^{\frac{b}{2}} \sqrt{\left(\frac{b \cdot L}{8 \cdot \Delta}\right)^2 + x^2}\, dx + \frac{(L - b) \cdot b}{2} \cdot \sqrt{\left(\frac{L}{4 \cdot \Delta}\right)^2 + 1}}$$

Mathcad Function:

$$q_{ef}(\Delta, D, L, b) := \left| \begin{array}{l} A \leftarrow \dfrac{\pi \cdot D^2}{4} \\ E_{eff} \cdot A \cdot \dfrac{2 \cdot \left[\displaystyle\int_{0 \cdot ft}^{\frac{b}{2}} \sqrt{1 + \left(\dfrac{8 \cdot \Delta \cdot x}{b \cdot L} \right)^2} \, dx + \sqrt{\left(\Delta - \dfrac{4 \cdot \Delta \cdot b}{L} \right)^2 + \left(\dfrac{L - b}{2} \right)^2} \right] - L}{2 \cdot \displaystyle\int_{0 \cdot ft}^{\frac{b}{2}} \sqrt{\left(\dfrac{b \cdot L}{8 \cdot \Delta} \right)^2 + x^2} \, dx + \dfrac{(L - b) \cdot b}{2} \cdot \sqrt{\left(\dfrac{L}{4 \cdot \Delta} \right)^2 + 1}} \end{array} \right.$$

Example 1: Given a 3/8" diameter 1x19 wire rope supported at 42", calculate the 1 foot length uniform load required to cause a maximum deflection of 1":

$D := 0.375 \cdot in$

$L := 42 \cdot in$

$b := 1 \cdot ft$

$\Delta := 1 \cdot in$

$q_{ef}(\Delta, D, L, b) = 158.4\,plf$

Example 2: Given a 3/8" diameter 1x19 wire rope supported at 42", calculate the 1 foot length uniform load required to cause a maximum deflection of 0.5":

$D := 0.375 \cdot in$

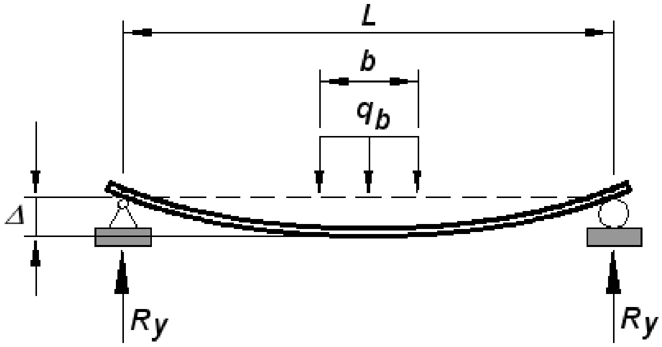
$L := 42 \cdot in$

$b := 1 \cdot ft$

$\Delta := 0.5 \cdot in$

$q_{ef}(\Delta, D, L, b) = 19.9\,plf$

C.2—LOAD-DEFLECTION RELATIONSHIP IN FLEXURAL BENDING



Symbols and Notations

Δ	Deflection of cable under uniform load, q_b, in.
b	Length of partial applied load, in.
D	Diameter of wire rope cable, in.
E_{eff}	Effective Modulus of Elasticity for wire rope cable, ksi.
I	Moment of Inertia, in^4.
I_{1x19}	Moment of inertia of 1x19 wire rope, in^4.
L	Spacing between intermediate supports, in.
q_b	Partial uniform load required to produce deflection Δ, lbs.

Objective

Given a deflection, Δ, determine the partial uniform load required to produce that deflection due to flexural bending.

Flexural Bending

The deflection of a simply-supported beam 3.5 feet long under a partial uniform load of length 1 foot was empirically computed to be:

$$\Delta = \frac{q_b \cdot b \cdot L^3}{50 \cdot E \cdot I}$$

This equation can be rearranged to calculate the load necessary to cause a deflection of Δ:

$$q_b = \frac{50 \cdot E \cdot I \cdot \Delta}{b \cdot L^3}$$

Mathcad Function:

$$q_b(\Delta, D, L, b) := \frac{50 \cdot E_{eff} \cdot I_{1x19}(D) \cdot \Delta}{b \cdot L^3}$$

Example: Given a 3/8" diameter 1x19 wire rope supported at 42", calculate the uniform load required to cause a deflection of 1" due to pure bending:

$$D := 0.375 \cdot \mathrm{in}$$

$$L := 42 \cdot \mathrm{in}$$

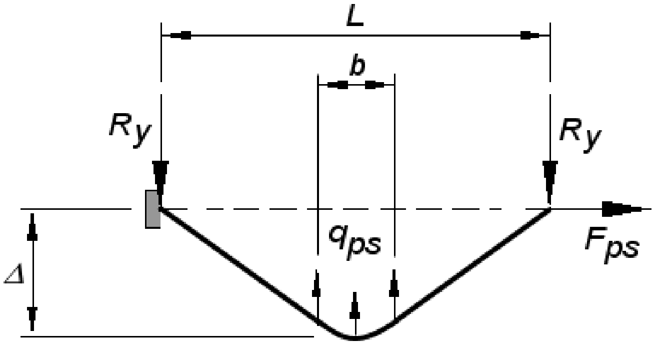
$$b := 1 \cdot \mathrm{ft}$$

$$\Delta := 1 \cdot \mathrm{in}$$

$$q_b(\Delta, D, L, b) = 7.998\,\mathrm{plf}$$

C.3—EFFECTS OF CABLE PRESTRESSING

The effect of cable prestressing is to provide a force to balance an applied load. This balancing force is directly related to the geometry of the cable and the prestressing force.



Symbols and Notations

Δ	Deflection of cable at mid-span, in.
b	Length of partial applied load, in.
L	Spacing between intermediate supports, in.
F_{ps}	Applied prestressing force, lbs.
q_{ps}	Partial uniform balance load due to prestressing, lbs.
R_y	Out-of-plane end reaction, lbs.

Objective

Given an initial prestress force, F_{ps}, and a mid-span deflection, Δ, determine the resulting balancing force, q_{ps}.

Balancing Force

The end reaction, R_y, can be found by taking the sum of the moments about the other end point:

$$\Sigma M = 0: \qquad q_{ps} \cdot b \cdot \frac{L}{2} - R_y \cdot L = 0$$

$$R_y \cdot L = \frac{q_{ps} \cdot b \cdot L}{2}$$

$$R_y = \frac{q_{ps} \cdot b \cdot L}{2}$$

Taking the sum of the moments at mid-span and considering forces to the right, we can compute the balance force q_{ps}:

Σ M = 0: $$q_{ps} \cdot \frac{b}{2} \cdot \frac{L}{4} - R_y \cdot \frac{L}{2} - F_{ps} \cdot \Delta = 0$$

$$\frac{q_{ps} \cdot b \cdot L}{4} - \frac{q_{ps} \cdot b \cdot L}{8} = -F_{ps} \cdot \Delta$$

$$q_{ps} = -\frac{8 \cdot F_{ps} \cdot \Delta}{b \cdot L}$$

Our applied load is equal to the magnitude of q_{ps}, but opposite in sign. Therefore, in the context of our applied load, the equation for q_{ps} becomes:

$$q_{ps} = \frac{8 \cdot F_{ps} \cdot \Delta}{b \cdot L}$$

Mathcad Function: $$q_{ps}(\Delta, L, b, F_{ps}) := \frac{8 \cdot F_{ps} \cdot \Delta}{b \cdot L}$$

Example: Given a 3/8" diameter 1x19 wire rope supported at 42", calculate the 1 foot long balancing load with a 400 lb prestress force and a deflection of 1":

$D := 0.375 \cdot in$

$L := 42 \cdot in$

$b := 1 \cdot ft$

$F_{ps} := 400 \cdot lbf$

$\Delta := 1 \cdot in$

$q_{ps}(\Delta, L, b, F_{ps}) = 76.2\ plf$

C.4—PUTTING IT ALL TOGETHER

Symbols and Notations

Δ	Deflection of cable under uniform load, *q.*
b	Length of applied partial uniform load, in.
D	Diameter of wire rope cable, in.
F_{ps}	Applied prestressing force, lbs.
L	Spacing between supports, in.
q	Partial uniform load required to produce deflection *Δ*, plf.
q_b	Component of uniform load, q, resisted by flexural bending, plf.
q_{ef}	Component of uniform load, q, resisted by stretching of cable, plf.
q_{ps}	Component of uniform load, q, resisted by cable prestressing, plf.

Combined Load-Deflection Relationship

The effects of cable stretch, flexural bending, and prestressing force combine to create a composite relationship between the applied load and the deflection of the cable. That is, for a given uniform load, the cable will deflect until the load is balanced by the sum of the reactions due to cable stretch, flexure, and prestressing force.

Recall the load-deflection relationships previously derived:

Extensible, Flexible Cable:

$$q_{ef} = E\cdot A\cdot \frac{2\cdot\left[\left[\int_0^{\frac{b}{2}} \sqrt{1+\left(\frac{8\cdot\Delta\cdot x}{b\cdot L}\right)^2}\,dx + \sqrt{\left(\Delta-\frac{4\cdot\Delta\cdot b}{L}\right)^2+\left(\frac{L-b}{2}\right)^2}\right]\right] - L}{2\cdot\int_0^{\frac{b}{2}} \sqrt{\left(\frac{b\cdot L}{8\cdot\Delta}\right)^2 + x^2}\,dx + \frac{(L-b)\cdot b}{2}\cdot\sqrt{\left(\frac{L}{4\cdot\Delta}\right)^2+1}}$$

Flexural Bending:

$$q_b = \frac{50\cdot E\cdot I\cdot\Delta}{b\cdot L^3}$$

Presstressing:

$$q_{ps} = \frac{8\cdot F_{ps}\cdot\Delta}{b\cdot L}$$

Strain compatibility laws tell us that when a load is applied to the cable, the deflection in each of the above cases must the same. Therefore, for a given deflection, the applied load required to cause that deflection is the sum of the three components:

$$q = q_{ef} + q_b + q_{ps}$$

Mathcad Function: $q(\Delta, D, L, b, F_{ps}) := q_{ef}(\Delta, D, L, b) + q_b(\Delta, D, L, b) + q_{ps}(\Delta, L, b, F_{ps})$

Example: Given a 3/8" diameter 1x19 wire rope supported at 42" and with a prestress load of 400 lbs., calculate the 1 foot long uniform load required to cause a mid-span deflection of 1":

$$D := 0.375 \cdot in$$

$$L := 42 \cdot in$$

$$b := 1 \cdot ft$$

$$F_{ps} := 400 \cdot lbf$$

$$\Delta := 1 \cdot in$$

$$q(\Delta, D, L, b, F_{ps}) = 242.6\,plf$$

C.5—BUILDING CODE LOAD REQUIREMENTS

The 2006 *International Building Code* and 2007 *California Building Code* require intermediate rails "to withstand a horizontally applied normal load of 50 pounds on an area equal to 1 square foot, including openings and space between rails" (IBC / CBC 1607.7.1.2).

To meet this requirement, the end reactions, R_x and R_y, caused by the 50 lbs over 1 square ft load must be determined, so that they may be included in the railing system frame calculations.

Symbols and Notations

Δ	Deflection of cable under uniform load, q_{app}.
b	Length of applied load q_{app}, in.
D	Diameter of wire rope cable, in.
F_{ps}	Applied prestressing force, lbs.
L	Spacing between supports, in.
q_{app}	Applied uniform load, plf.
R_x	In-plane support reaction, lbs.
R_y	Out-of-plane support reaction, lbs.

End Reactions

Recall, from Section C.1, that R_x and R_y can be found using the sum of the moments about one of the supports and about the middle of the cable, respectively. Substituting the applied partial load, q_{app}, for the partial load, q_{ef}, and noting that the deflection Δ represents the combined effects of the extensible-flexible cable, bending in the cable and prestressing, Eqns. (1) and (2) may be rewritten:

$$R_y = \frac{q_{app} \cdot b}{2} \qquad \textbf{(11)}$$

$$R_x = \frac{q_{app} \cdot b \cdot L}{8 \cdot \Delta} \qquad \textbf{(12)}$$

Determining the out-of-plane reaction, R_y, is straightforward; however, calculating the in-plane reaction, R_x, requires knowing the deflection in the cable, Δ. Given the applied uniform load, q_{app}, we can calculate the deflection using the Mathcad **root()** function introduced earlier:

$$\Delta = \text{root}\left(q\left(\Delta, D, L, b, F_{ps}\right) - q_{app}, \Delta\right)$$

Knowing the value for *Δ*, the reactions can be determined, as shown in the Mathcad functions below.

Mathcad Functions:

$$R_y\left(q_{app}, b\right) := \frac{q_{app} \cdot b}{2}$$

$$R_x\left(D, L, F_{ps}, q_{app}, b\right) := \left| \begin{array}{l} \Delta \leftarrow 0.1 \cdot \text{in} \\ \Delta \leftarrow \text{root}\left(q\left(\Delta, D, L, b, F_{ps}\right) - q_{app}, \Delta\right) \\ \dfrac{q_{app} \cdot b \cdot L}{8 \cdot \Delta} \end{array} \right.$$

Example. Given 3/8" wire rope cables spaced at 3.11", supported at 42", and with a 400 lb prestress force in each cable, determine the reactions for a single cable when subjected to the code-required 50 psf loading.

$D := 0.375 \cdot \text{in}$

$s := 3.11 \cdot \text{in}$

$L := 42 \cdot \text{in}$

$b := 1 \cdot \text{ft}$

$F_{ps} := 400 \cdot \text{lbf}$

$q_{app} := (50 \cdot \text{psf}) \cdot s$ $\quad q_{app} = 12.958\ \text{plf}$

$R_y\left(q_{app}, b\right) = 6.5\ \text{lbf}$

$R_x\left(D, L, F_{ps}, q_{app}, b\right) = 460.3\ \text{lbf}$

— END OF SECTION C —